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Atom Optics and Quantum Groups

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It is shown that in atom optics physical systems arise which have close similarities to quantum group structures. A particular example for which an atomic operator provides a representation of the quantum group $\mathrm{GL}_q(2, C)$ for $q = -1$ is presented.

1. INTRODUCTION

In the last decade considerable progress has been made in the field of atom optics (see, e.g., Adams *et al.*, 1994, and references therein). During the same period a new algebraical structure called a quantum group has gained much interest in mathematical physics (Drinfel'd, 1987; Fadeev *et al.*, 1988). Despite the huge amount of theoretical work concerned with this subject, there is, to the author's knowledge, only one application of this formalism to an experimentally accessible physical system. This is the description of molecular vibration and rotation spectra with the aid of a q-deformed Hamiltonian (Chang, 1995, and references therein). It is the purpose of this paper to argue that also in atom optics objects very similar to certain quantum groups naturally arise, and that in particular the special form of the interaction between atoms and laser fields allows one to identify certain operators as members of a special quantum group.

An attempt to connect quantum groups and quantum optics was previously done by Zhe (1992), who replaced ordinary photons by their q-deformed counterpart. Instead of introducing new q-deformed objects into physically established theories, it is the purpose of the present work to demonstrate that even nondeformed theories can show a quantum group structure.

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2. ATOM OPTICS

One of the main topics in atom optics is the influence of laser fields on the atomic center-of-mass motion. If the laser is nearly in resonance with an atomic transition it is often a very good approximation to take only two energy states, say $|g\rangle$ and $|e\rangle$, into consideration so that all operators are matrices with respect to the internal quantum states. The center-of-mass of motion is described by the position and momentum operators $\mathbf{x}$ and $\mathbf{p}$. A typical example of such a system is a two-level atom moving in a standing laser wave, for which the Hamiltonian is given by (see, e.g., Audretsch and Marzlin, 1994)

$$H = \frac{\mathbf{p}^2}{2M}\mathbf{1} - \frac{\hbar}{2}\Delta\sigma_3 - \frac{\hbar\Omega_0}{2}\cos(\mathbf{k}\cdot\mathbf{x})\,\sigma_1 \tag{1}$$

M is the mass of the atom, $\mathbf{1}$ denotes the unit matrix in two dimensions, and σ_i are the Pauli matrices. The quantity Ω_0 denotes the Rabi frequency, which is related to the laser field intensity. $\mathbf{k}$ is the wavevector of the laser. The detuning of the laser frequency ω_L versus the atomic transition frequency ω_{eg} is given by $\Delta := \omega_L - \omega_{eg}$. Spontaneous emission has been neglected.

To work out the similarity between such systems and quantum groups it is instructive to consider the resonant case $\Delta = 0$. In this case the Hamiltonian matrix can be diagonalized to the form

$$H = \frac{\mathbf{p}^2}{2M}\mathbf{1} - \frac{\hbar}{2}\Omega_0\cos(\mathbf{k}\cdot\mathbf{x})\,\sigma_3 \tag{2}$$

In the interaction picture with respect to the kinetic energy the Hamiltonian becomes

$$\begin{aligned} H' &= \tfrac{1}{4}\hbar\Omega_0\, e^{i\delta_r t}(e^{i\mathbf{k}\cdot\mathbf{x}}e^{it\mathbf{k}\cdot\mathbf{p}/M} + e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-it\mathbf{k}\cdot\mathbf{p}/M})\sigma_3 \\ &= \tfrac{1}{2}\hbar\Omega_0\cos(\mathbf{k}\cdot\mathbf{x} + t\mathbf{k}\cdot\mathbf{p}/M)\sigma_3 \end{aligned} \tag{3}$$

where $\delta_r := \hbar\mathbf{k}^2/(2M)$ is the recoil shift. In the derivation we have used, besides others, the relation

$$\exp(i\mathbf{k}\cdot\mathbf{x})\exp(it\mathbf{k}\cdot\mathbf{p}/M) = \exp(-2i\delta_r t)\exp(it\mathbf{k}\cdot\mathbf{p}/M)\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{4}$$

The occurrence of the expressions $\exp(i\mathbf{k}\cdot\mathbf{x})$ and $\exp(it\mathbf{k}\cdot\mathbf{p}/M)$ in the time evolution operator is a general feature of plane-wave laser fields. While the first of these exponentials stems from the spatial variation of the monochromatic laser wave, the second one is a consequence of the Doppler shift of the laser's frequency in the atomic reference frame. Note that equation (4) provides a representation of the Weyl algebra (see, e.g., Barut and Racka, 1977).

3. QUANTUM GROUPS

A well-known example of a quantum group is the system of 2×2 matrices M with

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \begin{array}{lll} ba = qab, & db = qbd, & [b, c] = 0 \\ ca = qac, & dc = qcd, & [a, d] = (q^{-1} - q)bc \end{array} \tag{5}$$

(q is a complex number). These relations define the quantum general linear group $\mathrm{GL}_q(2, C)$. In general, quantum groups are noncommutative Hopf algebras which are derived from commutative ones by means of an algebra deformation. For our purposes it is sufficient to know that a Hopf algebra $\mathcal{H}$ is a space with a multiplication m: $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and a comultiplication Δ: $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ which is a multiplication in dual space [for details on quantum groups see Doebner *et al.* (1990), for instance]. For a Hopf algebra, m and Δ must be compatible in the sense that

$$[\Delta(a), \Delta(b)] = \Delta([a, b]) \qquad \forall a, b \in \mathcal{H} \tag{6}$$

holds. In this equation the commutators are defined by using the multiplication m. This compatibility of multiplication m and comultiplication Δ is one of the essential features of a quantum group. In the example (5) the deformed algebra is the algebra generated by the matrix components a, b, c, and d and the multiplication m is defined by the commutation relations given in (5). The comultiplication Δ is related to the matrix multiplication and can be written in the form

$$\Delta(M) = M \otimes M \tag{7}$$

For a single matrix component this reads $\Delta(a) = a \otimes a + b \otimes c$ and so on. In the limit $q \to 1$ the original general linear group $\mathrm{GL}(2, C)$ with commuting matrix components is recovered.

4. CONNECTIONS BETWEEN BOTH FIELDS

The idea that certain systems in atom optics may show the structure of a quantum group is based on two observations. The first is that in both fields noncommuting matrix components naturally arise; compare equations (5) and (1), for instance. This resemblance is of course not a special feature of atom optics, since operator-valued matrix components also occur in other fields, e.g., in the field equations for Dirac or Pauli spinors.

The second similarity between atom optics and quantum groups is more characteristic than the first one. It is based on the commutators between the matrix components given in equation (5). It is typical for these commutators that the interchange of a and b, say, results in the multiplication with a complex number q. But this is also a feature of the Weyl algebra of equation (4), which naturally arises in atom optics. The apparent similarity between both relations with q being replaced by $\exp(-2i\delta_r t)$ is a further connection between atom optics and the quantum group $GL_q(2, C)$.[2] While the first observation was based on the matrix structure and therefore on the comultiplication Δ, the second observation is a statement about the commutators of the matrix components and therefore concerns the multiplication m.

Although it now seems to be easy to construct an atomic operator which may be interpreted as a representation of the quantum group $GL_q(2, C)$, this task turns out to be surprisingly difficult. The reason is that besides the operators which represent the Weyl algebra (4) there are a couple of other operators, like the kinetic energy, for instance, which have to be taken into account. Nevertheless, there is at least one example for which the correspondence can be established. It is provided by a simple one-dimensional model for a two-level atom with very high velocity in a running laser wave. The Hamiltonian can be deduced from equation (1) by replacing the kinetic energy by cp and the standing laser wave part $\cos(kx)$ by $\exp(ikx)$. After a unitary transformation with the operator O given by $O_{11} = \exp(ikx/2) = O_{22}^{+}$ the Hamiltonian $\tilde{H} = O^{+}HO$ becomes

$$\tilde{H} = \frac{\hbar}{2}(\omega_{eg}\sigma_3 - \Omega_0\sigma_1) + cp\mathbf{1} \tag{8}$$

The time evolution operator $\tilde{U} = \exp(-i\tilde{H}/\hbar)$ is not difficult to obtain from this expression. Setting $S := \exp[-it(\omega_{eg}\sigma_3 - \Omega_0\sigma_1)/2]$ (the precise form of S can be easily derived, but is unimportant in what follows), we can write the total evolution operator as

$$U = \tilde{U}\tilde{O}O^{+} = e^{-icpt/\hbar}\begin{pmatrix} S_{11}e^{it\omega_L/2} & S_{12}e^{it\omega_L/2}e^{ikx} \\ S_{21}e^{-it\omega_L/2}e^{-ikx} & S_{22}e^{-it\omega_L/2} \end{pmatrix} \tag{9}$$

To demonstrate that the operator (9) is related to the quantum group $GL_q(2, C)$ at certain moments of the time evolution, the matrix components of equation (9) are denoted by α, β, γ, and δ and are shown to fulfill

$$\begin{aligned} &\beta\alpha = \epsilon\alpha\beta, \qquad \gamma\alpha = \epsilon^{-1}\alpha\gamma, \qquad \alpha\delta = \delta\alpha \\ &\gamma\beta = \epsilon^{-1}\beta\gamma, \qquad \delta\beta = \epsilon^{-1}\beta\delta, \qquad \delta\gamma = \epsilon\gamma\delta \end{aligned} \tag{10}$$

where $\epsilon := \exp(i\omega_L t)$. These relations agree with the commutation relations

[2] Besides this fact, the operators $\exp(i\mathbf{k}\cdot\mathbf{x})$ and $\exp(it\mathbf{k}\cdot\mathbf{p}/M)$ may be of mathematical interest in themselves because relations like $[\sin(\mathbf{k}\cdot\mathbf{x} + t\mathbf{k}\cdot\mathbf{p}/M), \cos(\mathbf{k}\cdot\mathbf{x} + t\mathbf{k}\cdot\mathbf{p}/M)] = 0$ and $\cos(\mathbf{k}\cdot\mathbf{x} + t\mathbf{k}\cdot\mathbf{p}/M) = \exp(i\delta_r t)[\cos(\mathbf{k}\cdot\mathbf{x})\cos(t\mathbf{k}\cdot\mathbf{p}/M) - \sin(\mathbf{k}\cdot\mathbf{x})\sin(t\mathbf{k}\cdot\mathbf{p}/M)]$ provide a kind of noncommutative trigonometry.

(5) if $-q = \epsilon^2 = 1$. This is the case at times $t_n := n\pi/\omega_L$, where n is an integer number. If n is even, we have $\epsilon = 1$ and all matrix elements commute. But for odd n the commutation relations (10) are nontrivial and provide a realization of the quantum group $GL_q(2, C)$ for $q = -1$ in atom optics.

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